exercise:

① check $\frac{6i}{2+\sqrt{2}i} = \sqrt{2}+2i$ by using exponential form on the left.

② Find the three zeros of polynomial z^3+8, and show that

$z^3+8=(z^2-2z+4)(z+2)$

③ show that the set containing $\exp(i\frac{-2k\pi}{m})$, $k=0,1,\cdots,m-1$ is the same as the set $\exp(i\frac{2k\pi}{m})$, $k=0,1,\cdots,m-1$. (up to order)

④ use def to show $\lim_{z\to 0}\frac{\bar{z}^2}{z}=0$

Solutions:

① $6i: r=\sqrt{0+6^2}=6$

$\theta=\sin^{-1}(1)=\frac{\pi}{2}$

$6i=6e^{i\frac{\pi}{2}}$

$2+\sqrt{2}i:\ r=\sqrt{2+4}=\sqrt{6}e^{i\theta}$

$\sin\theta=\frac{\sqrt{2}}{(\sqrt{2}^2+2^2)^{\frac{1}{2}}}=\frac{1}{\sqrt{3}}$

$\cos\theta=\frac{2}{(\sqrt{2}^2+2^2)^{\frac{1}{2}}}=\frac{2}{\sqrt{6}}$

$\frac{6i}{2+\sqrt{2}i}=\frac{6e^{i\frac{\pi}{2}}}{\sqrt{6}e^{i\theta}}$

$=\sqrt{6}e^{i(\frac{\pi}{2}-\theta)}$

$\xrightarrow{euler}=\sqrt{6}\sin\theta+i\sqrt{6}\cos\theta$

$=\sqrt{2}+2i$

③ using $e^{2\pi i}=1$

$e^{(i\cdot-\frac{2k\pi}{m})}=e^{(i\cdot-\frac{2k\pi}{m})}\cdot e^{2\pi i}$

$=e^{(i\cdot\frac{2(m-k)\pi}{m})}$

$k=0,1,\cdots,m-1$

$\rightarrow m-k=1,2,\cdots,m-1,m$

Let's make $t=m-k$

$e^{(i\frac{2(m-k)\pi}{m})}=e^{i\frac{2t\pi}{m}}$, $t=1,2,\cdots,m$

$=0,1,\cdots,m-1$

($e^{i\frac{2t\pi}{m}}=1$ when $t=0$ or m)

② $z^3+8=0$

$z^3=-8=8e^{i(\pi+2k\pi)}$

$C_k=\sqrt[3]{8e^{i(\pi+2k\pi)}}$

$=2e^{i(\frac{\pi}{3}+\frac{2}{3}k\pi)}$

$C_0=1+\sqrt{3}i$ $(k=0)$

$C_1=-2$ $(k=1)$

$C_2=1-\sqrt{3}i$ $(k=2)$

$z^3+8=(z-C_0)(z-C_1)(z-C_2)$

$=(z^2-2z+4)(z+2)$

④ to prove a limit ($\lim_{z\to 0}\frac{\bar{z}^2}{z}=0$), we need for any $\varepsilon>0$, there exist $\delta>0$, s.t.

$0<|z|<\delta\Rightarrow|\frac{\bar{z}^2}{z}-0|<\varepsilon$

$z=x+yi$, $\bar{z}=x-yi\Rightarrow|\bar{z}|=|z|=\sqrt{x^2+y^2}$

So $|\frac{\bar{z}^2}{z}|=\frac{|\bar{z}^2|}{|z|}=\frac{|z^2|}{|z|}=|z|$

to ensure $|z|<\varepsilon$, we set,

$\delta=\varepsilon$

when $0<|z|<\delta$, $|z|<\varepsilon$.